# Oblique derivative problem for elliptic second-order equations in a domain with boundary conical point

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## SUMMARY

Linear second-order elliptic equations on bounded smooth domains have been studied from the first half of the twentieth century. Works of Giraud and Schauder in the thirties showed that, assuming sufficient smoothness of the coefficients and of the boundary of the domain, the basic boundary-value problems are solvable. The article of Friedrichs in 1934 initiated an approach in which they were interpreted from the standpoint of functional analysis. The articles by De Giorgi in 1957 and Nash in 1958 prompted a start of a new stage in the study of many-dimensional linear equations. The culmination of this stage were works of Stampacchia (1960), Morrey (1959), and Ladyzhenskaya and Ural'tseva (1964). Their research inspired a number of other mathematicians.

Problems in which the boundary value condition has the form $b(x, u, Du) = 0$, where $b$ depends on the gradient $Du$ of the unknown function $u$ in a suitable way, are called oblique derivative problems. It is worth noting that there was a major difference in attitudes towards oblique and Dirichlet problems in the eyes of most researchers in partial differential equations. Ladyzhenskaya and Ural'tseva in their book devoted nine chapters to the Dirichlet problem and only one (Chapter 10) to "other boundary problems". Gilbarg and Trudinger in their book devoted only several pages to oblique derivative problem. The systematic development of the theory of oblique derivative problems for elliptic equations was presented in 2013 by Lieberman. Unfortunately, the theory of oblique derivative problem is not yet in the complete state, like it is for Dirichlet problem.

It should be noted that investigations in the aforementioned works refer to boundary value problems in sufficiently smooth domains. However, many problems of physics and technology lead to the necessity of studying boundary value problems in domains with non-smooth boundary, in particular, in domains which have a finite number of angular ($n = 2$) or conical ($n \geq 3$) points on the boundary. The theory of boundary value problems in non-smooth domains was described in well-known survey of Kondratiev and Oleinik (1983) and in the work of Kufner and Sändig (1987), as well as in the monographs of Maz'ya et al. (1984, 1997).

The pioneering and fundamental works devoted to the general linear boundary value problems for domains with angular or conical pints were the papers of Kondratiev (1963, 1967), Birman

and Skvortsov (1962), Eskin (1963, 1985), Lopatinskiy (1963), and Maz'ya (1963, 1966, 1967). These works are related to normal solvability and regularity for general linear elliptic problems in the weighted Sobolev-Kondratiev spaces in non-smooth domains under sufficient smoothness of the equations coefficients. It is worthy of note that the methods used for analysis problems in smooth domains cannot be applicable to the analysis of problems in non-smooth domains.

This dissertation is devoted to investigation of the behaviour of strong solutions of the oblique derivative problem for elliptic equations in a neighborhood of a boundary conical point. The author's main goal is to establish the sharp exponent of the solution decrease rate under the best possible conditions. The theory of oblique derivative problem has played a major role in the study of reflected shocks in transonic flow. Another important application of this theory is the capillary problem. The description of this theory, as it was known in 1986, can be found in a book by Finn (1986). In the science of construction materials, singular points correspond to the properties of materials near the cracks or fissures. Some applications of boundary value problems, both linear and nonlinear, in composite materials with a finite number of inclusions are presented in the monograph by Mityushev and Rogosin (1999). In geodesy some of the most fundamental problems of the gravity field determination from boundary observations are translated into exterior boundary value problems for the Laplace or Poisson equation.

In the dissertation we study the behavior of strong solutions to the oblique derivative problem for non-divergent: linear, semi-linear and quasi-linear second-order elliptic equations in an $n$-dimensional bounded domain with a boundary conical point. We derive *a priori* estimates in the Sobolev-Kondratiev spaces and estimates of the type $u(x) = O(|x|^\alpha)$ with sharp exponents $\alpha$.

We also derive the Friedrichs-Wirtinger type inequality adapted to the linear problem with an sharp estimating constant and establish some auxiliary integro-differential inequalities. We derive for solutions local and global weighted estimates and find the best exponents of the continuity power modulus at the conical boundary point. We consider the solution estimates for equations with minimal smooth coefficients that is a principal new feature of the work. We derive some estimations of solutions in the case, when the equations coefficients do not satisfy the Dini-continuity condition.

We prove the theorem that there exists the smallest positive eigenvalue of the eigenvalue problem for the Laplace-Beltrami operator on the unit sphere. We also derive the estimation of the eigenvalue.

Obtained results are extension of the works of Borsuk and Kondratiev, and the previous work of the author (*Electronic Journal of Differential Equations*, 2012). Partial results presented in the dissertation were published in three papers: two in *Complex Variables and Elliptic Equations* (2014, 2015) and one in *Current Trends in Analysis and Its Applications, Trends in Mathematics* (2015).

In this thesis we applied the Borsuk-Kondratiev methods that are adapted to the oblique derivative problems in non-smooth domains. We also use Sobolev imbedding theorems, $L^p$-estimates, maximum principle theorems, comparison principle theorems as well as the method of the barrier function. In the proof of the existence theorem we use the Legendre spherical harmonics, Gegenbauer functions, and some properties of these functions. To derive the estimations of the smallest positive eigenvalue we use the Chaplygin maximum principle.